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LETTER TO THE EDITOR

Subtle dynamic behaviour of finite-size Sherrington–Kirkpatrick spin glasses with non-symmetric couplings

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Abstract. We have studied numerically the parallel dynamics of non-symmetric Sherrington–Kirkpatrick spin glasses, varying the *degree of symmetry* $\eta := \langle J_{i,k}J_{k,i}\rangle/\langle J_{i,k}^2\rangle$ of the coupling coefficients between 0 and 1. For systems of *finite* size N, in the limit $t \to \infty$ and at 'zero temperature', $T = 0$, we find subtle behaviour of the function $\langle C_2(t)\rangle := \langle s_i(t-1)s_i(t+1)\rangle$, which characterizes the appearance of 2-cycles or fixed-point attractors quantitatively.

One has to distinguish the two cases

- $\eta > 0.5$, where $\langle C_2(\infty)\rangle \to 1$, i.e. the system is eventually trapped with probability 1 in a fixed-point or a 2-cycle, if after $t \to \infty$ the limit $N \to \infty$ is taken, and
- $\eta < 0.5$, where, in contrast, $\langle C_2(\infty)\rangle$ is < 1, since longer cycles appear.

However, the 'trapping' for $\eta > 0.5$ happens only for $T = 0$, and at time scales τ_N which increase exponentially with N, whereas for $T > 0$, or if for $T = 0$ the limit $N \to \infty$ would be taken *before* $t \to \infty$, the quantity $\langle C_2(t \to \infty)\rangle$ would decrease smoothly and monotonically with decreasing η right from $\eta = 1$, in quantitative agreement with the mean-field simulation of Eissfeller and Opper.

For $T = 0$, the transient behaviour of $\langle C_2(t)\rangle$ between the mean-field value, which is reached already after typically 100 time steps, and the trapping event, is found to be to be governed by log–normal statistics with size-depending parameters.

The *parallel* dynamics of Sherrington–Kirkpatrick spin glasses with non-symmetric couplings have been treated in a number of recent papers [1, 2] as an important example of a complex dynamic system, where due to the lack of a *Hamiltonian* the attractors can be complicated: e.g. in addition to fixed point attractors *cycles* can also appear, and the periods and transient times of these cycles can be very large. In fact, in numerical experiments, one of the present authors [3] has found that below a *critical value* $\eta_c \approx 0.5$ of the symmetry parameter η to be defined in (3), cycles exist, with periods diverging exponentially with the system size N.

In the present letter this problem is studied in detail, i.e. we concentrate on the *long-time* behaviour of the so-called '2-cycle function' $\langle C_2(t)\rangle := \langle s_i(t-1)s_i(t+1)\rangle$. Here $s_i = \pm 1$ are the variables of the system, i the sites, t the (integer) time, and $\langle\cdots\rangle$ means an average over i as well as over various samples and initial conditions (see below). Obviously, $\langle C_2(\infty)\rangle = 1$ would mean that with probability 1 the attractor of the system has either period 1 (= fixed-point) or period 2 (= two-cycle).

As a result of extensive numerical simulations we find below that at $T = 0$, i.e. without thermal noise, the behaviour of $\langle C_2(t)\rangle$ is quite subtle, depending on whether η is larger than 0.5 or not.

If η is > 0.5, one has $\langle C_2(t)\rangle \to 1$ if $t \to \infty$ for large fixed N, or more precisely: if N is sent $\to \infty$ *later* than $t \to \infty$. To be precise this means that for a finite system at $T = 0$ and $\eta > 0.5$ the system is trapped either in a fixed-point or a 2-cycle at a time τ_N, which depends on the sample considered and on the initial conditions. As discussed below, typical values of τ_N increase exponentially with N.

In contrast, as long as the 'thermal noise' T is large enough, or at $T = 0$, if $N \to \infty$ is taken *before* $t \to \infty$, then $\langle C_2(t)\rangle$ saturates at a finite value < 1, which for sufficiently large N does not depend on N and decreases smoothly with decreasing η right from $\eta = 1$, in quantitative agreement with the mean-field calculation of Eissfeller and Opper [1].

Thus our results shed light on the subtle problem of the *interchange of limits* $t \to \infty$, $N \to \infty$ and $T \to 0$ in the present non-symmetric Sherrington–Kirkpatrick spin glass system with parallel dynamics, where the behaviour depends on several time-constants (see below). For *random sequential dynamics* similar subtleties concerning the interchange of limits have already been found by Crisanti and Sompolinsky [4] (see the discussion below), although for those dynamics the physics is simpler, i.e. cycles cannot appear and at $\eta = 0.5$ nothing special happens.

We consider a system of sites $i = 1, \ldots, N$ with Ising spin degrees of freedom, $s_i = \pm 1$, which can e.g. be considered as the two states of formal neurons, $s_i = +1$ representing the 'firing' and $s_i = -1$ the 'passive' state. The dynamics of the system are described at $T = 0$ by

$$s_i(t+1) = \mathrm{sign}\left[\sum_{k(\neq i)=1}^{N} J_{i,k} s_k(t)\right] \tag{1}$$

where the updating is performed simultaneously for all sites i ('parallel dynamics').

For finite 'temperature', $T > 0$, equation (1) is replaced by the probabilistic rule

$$\mathrm{prob}[s_i(t+1) = \pm 1] = [1 + \exp(\mp 2h_i(t)/T)]^{-1} \tag{2}$$

where $h_i = \sum_{k(\neq i)} J_{i,k} s_k$. (It should be noted that in (2) the probilities are already correctly normalized: $0 \le \mathrm{prob}[s_i] \le 1$ and $\mathrm{prob}[s_i = +1] + \mathrm{prob}[s_i = -1] = 1$.)

It is well known that the dynamics of this system, which can be solved analytically only for the special case of complete asymmetry $\eta = 0$ (see below), both for parallel dynamics [2], and also for random sequential dynamics [4, 5] depend crucially on the symmetry parameter η of the couplings $J_{i,k}$, which are generated as follows. One chooses independent Gaussian random numbers x and y with average 0 and variance 1 and defines $J_{i,k} = (x + \rho y)/\sqrt{N(1+\rho^2)}$ and $J_{k,i} = (x - \rho y)/\sqrt{N(1+\rho^2)}$. Then one has $\langle J_{i,k}\rangle = \langle J_{k,i}\rangle = 0$ and $\langle J_{i,k}^2\rangle = \langle J_{k,i}^2\rangle = 1/N$, and the symmetry parameter is

$$\eta := \frac{\langle J_{i,k} J_{k,i}\rangle}{\langle J_{i,k}^2\rangle} = \frac{1-\rho^2}{1+\rho^2}. \tag{3}$$

Here we concentrate on parallel dynamics, since the simpler case of random sequential dynamics has already been studied in [4] and the case of deterministic sequential updating in [6]. For different values of the *symmetry parameter* η, starting in each case with typically 1000 random samples, where both the couplings and the input states were randomly chosen, we have iterated the dynamical equations (1) and (2) up to very long times, e.g. $t = 20\,000$,

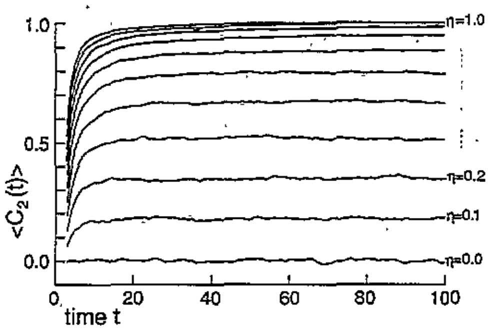

Figure 1. The averaged '2-cycle function' $\langle C_2(t)\rangle := \langle s_i(t-1)s_i(t+1)\rangle$ is presented over the number t of time steps of the dynamics of (1) with parallel updating, averaged over 100 random samples and $N = 640$ different sites i per sample of a completely connected non-symmetric Sherrington–Kirkpatrick spin glass model with fixed values of the symmetry parameter η defined in (3), ranging from the completely asymmetric case of $\eta = 0$ up to the symmetric case of $\eta = 1.0$. For every sample, a random initial condition has been chosen.

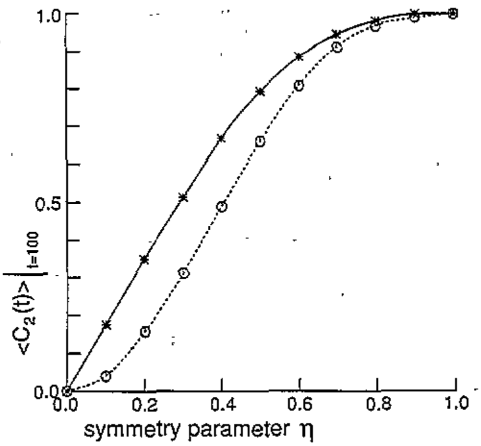

Figure 2. The dependence of the apparent *plateau values* C_{MF} reached in figure 1 at $t = 100$ on the symmetry parameter η is presented by crosses joined by the solid line, showing the behaviour predicted by the molecular field theory of [1] for parallel updating. For comparison we also present similar results obtained by [6] for deterministic sequential updating, plotted as open circles joined by a dotted line. The error bars are smaller than the symbol size.

studying the expectation value $\langle C_2(t)\rangle := \langle s_i(t-1)s_i(t+1)\rangle$ as a function of t in the limit of large t for different values of N, η and T.

In the following, we present and discuss our results.

In figure 1, for $T = 0$ and $\eta = 0, 0.1, 0.2, \ldots, 1.0$ we present the average $\langle C_2(t)\rangle$ over t in a 'short-time' regime $0 < t < 100$ for $N = 640$, where averages over all 640 sites for 100 samples have been performed (1000 samples have been used in the long-time studies to be discussed later). According to the results of figure 1, already after ≈ 10 to ≈ 40 time steps (the actual number increases gradually with η), $\langle C_2(t)\rangle$ seems to saturate at a value, which is < 1 as long as $\eta < 1$. The fluctuations around the average $\langle C_2(t = 100)\rangle$ in figure 1 are finite size effects and smaller than $\approx 0.5 \times 10^{-2}$. This is just what one expects from the statistics, since one is dealing with results from 100 random samples and 640 sites, so that the fluctuation amplitude should be $\leq 64\,000^{-1/2} \approx 0.4 \times 10^{-2}$.

For $\eta = 1$, Gardner *et al* [7] have shown that the quantity presented in figure 1 converges to 1 according to a power law $1 - \langle C_2(t)\rangle \propto t^{-1.5}$. Although this is not a major point of our study, we mention that a detailed analysis of figure 1 for short times shows that also for $\eta < 1$ the results of figure 1 apparently converge to the plateau value $\langle C_2(t = 100)\rangle$ with this power law, i.e. $\langle C_2(100)\rangle - \langle C_2(t)\rangle \approx c(\eta)t^{-1.5\pm0.01}$, where the prefactor $c(\eta)$ decreases with decreasing η. However, the range of t, where our results are statistically significant enough to ascertain this power law (which is the case as long as $\langle C_2(100)\rangle - \langle C_2(t)\rangle$ is still larger than ≈ 0.05), shrinks with decreasing η from $t \leq 12$ for $\eta = 0.6$ down to $t \leq 4$ for $\eta = 0.1$. We do not present an extra plot of the detailed behaviour, since it is quite similar to what Pfenning *et al* [6], have observed and plotted for sequential dynamics (see figure 2 of [6], where only the prefactors are roughly half as large as ours). However the saturation value itself, which they obtain for given η with deterministic sequential dynamics, differs quantitatively and qualitatively from what we get with deterministic parallel dynamics, as can be seen from figure 2.

Figure 2 presents the apparent plateau values, as obtained from our figure 1, as a function of η. The errors in this figure, as mentioned above, are smaller than the symbol size. The results agree quantitatively with the recent results obtained by Eissfeller and Opper [1] for $t \to \infty$ in a numerical implementation of the mean-field equations for parallel dynamics, which are exact if the limit $N \to \infty$ is taken first. These results will be referred to as 'mean-field limits', C_{MF}, below; for sufficiently large N they do not depend on N.

As already mentioned, the results of Pfenning *et al* [6] for sequential dynamics are also plotted in figure 2. Interestingly, with sequential instead of parallel dynamics, the effect of the 'asymmetry noise' is stronger, since the plateau values $\langle C_2(t = 100)\rangle$ of [6] are significantly smaller, particularly for $\eta < 0.2$ they behave $\propto \eta^2$, whereas in our case of parallel dynamics the increase is simply $\propto \eta$. We do not yet have an explanation for this difference at present.

Now we come to our main point. In our case, where one is dealing with *finite* systems, the mean field theory for parallel dynamics of Eissfeller and Opper [1] is no longer applicable, if $t \to \infty$ is taken first. In fact, extending our calculations to the much longer time scales mentioned above we find the behaviour presented in figure 3. There, for $\eta = 0.6$ and $T = 0$, averaging over all sites and now over 1000 samples, we see that the averaged $\langle C_2(t)\rangle$, for t larger than a second relaxation time τ_N, always ends at a finite plateau value $C_\infty(N)$ which considerably exceeds the 'mean-field' limit C_{MF} mentioned above, e.g. $C_\infty(N)$ is as large as 0.97 already for $N = 16$, whereas $C_{\mathrm{MF}} = 0.88$ for the η value considered ($\eta = 0.6$), see figure 2. Moreover, according to figure 3, $C_\infty(N)$ converges to 1 for $N \to \infty$, which means that in this limit all samples are eventually *trapped* in a 2-cycle or fixed point. However, the characteristic times τ_N, where the plateau value $C_\infty(N)$ of $\langle C_2(t)\rangle$ is reached, obviously also increase drastically with N, e.g. from figure 3 it can be seen that $\tau_N \approx 20\,000$ for $N = 256$. This will be discussed more in detail below.

These results, which also have been obtained for other values of η between 0.5 and 1, mean that for $t > \tau_N$, if the limit $N \to \infty$ is taken under this constraint, i.e. after $t \to \infty$, then the systems with $T = 0$ and $\eta > 0.5$ get stuck either in a fixed point or in a 2-cycle with probability 1. (The case of $\eta < 0.5$, which has already been studied by one of us in a former paper [3], is mentioned further below.)

In figure 4, again for $\eta = 0.6$ we study how this 'trapping' behaviour is changed by a small finite temperature for three sample sizes, $N = 128$, $N = 192$ and $N = 256$. There, up to $t = 2500$, the probabilistic dynamics of (2) have been chosen, with a small temperature of $T = 0.04$. From figure 4 it can be seen that for $N = 192$ and $N = 256$ even this low temperature is high enough to prevent the above-mentioned *trapping* in most cases, so that for these values of N, the sample-average $\langle C_2(t)\rangle$ is essentially constant < 1 already for $t > 100$, whereas for $N = 128$, T is not high enough to prevent the gradual increase of the average $\langle C_2(t)\rangle$, i.e. to prevent the trapping of more and more samples with increasing time. Thus the necessary temperature for the suppression of trapping increases with decreasing N.

In any case, switching off the thermal noise for $t > 2500$, one finds in the right part of figure 4 that for all three values of N the function $\langle C_2(t)\rangle$ converges monotonically to $\langle C_2(\infty)\rangle = 1$, i.e. for $T = 0$ and $\eta > 0.5$ we find that trapping cannot be avoided, although the characteristic trapping time increases drastically with N. However, one should note at this place that even such large trapping times as $\tau_N \approx 10\,000$ for $N \approx 200$ would still be much less than the ultimate time $\tau_{\mathrm{repeat}} = 2^N$ ($\approx 10^{60}$ for $N \approx 200$), after which the state

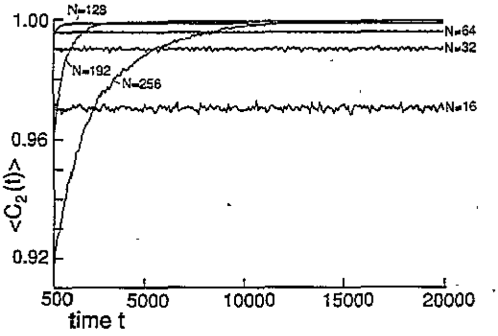

Figure 3. The long-time behaviour of the averaged '2-cycle function' $\langle C_2(t)\rangle$ is presented over the time for up to 20 000 time steps, averaged for 1000 samples for fixed size of the system, ranging from $N = 16$ to $N = 256$, for $\eta = 0.6$.

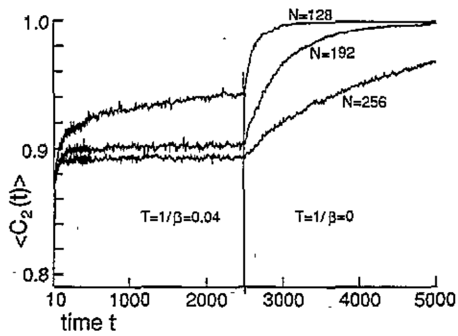

Figure 4. The long-time behaviour of $\langle C_2(t)\rangle$ is presented over t for the systems of figure 3 for $N = 128$, $N = 192$ and $N = 256$. Between $t = 1$ and $t = 2500$, the stochastic parallel dynamics of (2) have been applied with a low 'temperature' $T = 0.04$, whereas at $t = 2500$ the system has switched to deterministic updating, i.e. with $T = 0$, equation (1). For further explanation see the text.

of the finite system definitely must repeat, if it has not done so before.

To understand the N-dependence of the necessary temperature, we have studied the behaviour of single typical samples. Figure 5 shows the typical behaviour of $\langle C_2(t)\rangle$ for two system sizes, $N = 256$ and $N = 1024$. For $N = 256$, $\langle C_2(t)\rangle$ at first fluctuates strongly around the 'mean-field limit' C_{MF}, before eventually a strong-enough fluctuation leads to a sudden trapping (see the inset in figure 5(a), whereas for $N = 1024$ the fluctuations around the average are reduced by a factor 1/2, i.e. the fluctuation amplitude seems to decrease $\propto N^{-1/2}$ as expected, so that for $N = 1024$ the system does not get trapped on

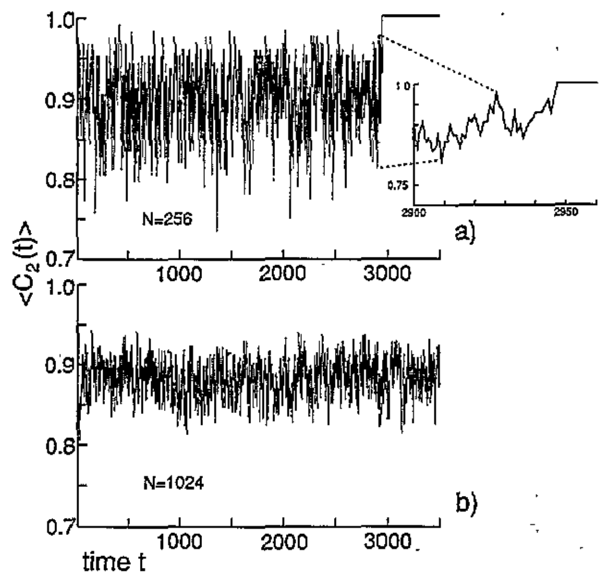

Figure 5. The typical behaviour of $\langle C_2(t)\rangle$ is presented, where only the average over all sites $i = 1, \ldots, N$ of one random sample has been performed, with $N = 256$ in the upper part (a), and $N = 1024$ in the lower part (b), for $\eta = 0.6$. In the inset, the time interval where the system is trapped in the '2-cycle attractor' at $t = 2946$ in (a) is presented in a magnified form. Note that in (b) the fluctuations are reduced by a factor of 0.5, i.e. their amplitude is $\propto N^{-1/2}$.

the time-scale considered.

Thus for $\eta > 0.5$ at $T = 0K$ only fixed points or cycles with period 2 appear as traps in the limit '$t \to \infty$ first and then $N \to \infty$'.

On the other hand, for $\eta < 0.5$, too, we find 2-cycle traps with relatively high probability. However, now we also find attractors with very long period, where even if the system is trapped in such an attractor, $\langle C_2(t)\rangle$ simply varies around the 'mean field' average $C_{\mathrm{MF}}(\eta)$ presented in figure 2. The probability to find such a long-period attractor decreases with increasing period, but *increases with decreasing* η, as can be seen from figure 5 and figure 6 of [3] already mentioned. These 'long-period attractors' are, however, very sensitive to small changes of the coupling constants: e.g., for systems with $N \simeq 150$ and $\eta = 0.4$, changes as small as $\Delta\rho \approx 10^{-4}$ (see equation (3)) already lead to the decay of a long attractor into shorter attractors in the vicinity of the orbit of the original long cycle.

At this point the following remark is in order. From our limited number of simulations we can of course fix the apparent 'critical value' of η only approximately, namely to $\eta_c \approx 0.5 \pm 0.025$, since in figure 2 of [3], where the range of η has been covered rather completely, we have already found the exponential increase of the cycle time with N clearly for $\eta = 0.45$ and below, but clearly not for $\eta = 0.55$ and above.

An interesting question is the dependence of the trapping time τ_N on N for our Sherrington–Kirkpaptrick spin glass with non-symmetric couplings and parallel dynamics. In their study for a similar model with *random-sequential dynamics*, Crisanti and Sompolinsky [4] have found 'log-normal behaviour', i.e. a normal distribution for $\ln \tau_N$, with average and mean-square deviation increasing linearly with N for $\eta = 0.72$ and $\eta = 0.47$. Such a behaviour is in agreement with our results for *parallel dynamics*, as can be seen from figure 6, where we also find a 'log-normal distribution' for the different τ_N obtained from the different samples used for $N = 256$ in figure 4.

From this 'log-normal' distribution, the results for $N = 256$ in figure 4 can simply be obtained by successively 'filling the distribution', i.e. with

$$p(\tau_N)\mathrm{d}\tau_N \simeq (2\pi\Delta_N^2)^{-1/2} \exp\left\{-\frac{[\ln \frac{\tau_N}{\tau_0(N)}]^2}{2\Delta_N^2}\right\} \mathrm{d}\left[\ln \frac{\tau_N}{\tau_0(N)}\right] \qquad (4)$$

one has

$$\langle C_2(t)\rangle \simeq C_{\mathrm{MF}} + (1 - C_{\mathrm{MF}}) \int_{\tau_{\mathrm{MF}}}^{t} p(\tau_N)\,\mathrm{d}\tau_N\,. \qquad (5)$$

Here the time $\tau_0(N)$ appearing in the distribution is roughly the 'halfway time' between $\tau_{\mathrm{MF}} \approx 100$, where in figure 2 the 'mean-field' saturation value C_{MF} of $\langle C_2(t)\rangle$ is reached, and the final trapping at $\langle C_2(t)\rangle = 1$, i.e. from figure 2 for $\eta = 0.6$ one has $C_{\mathrm{MF}} = 0.885$,

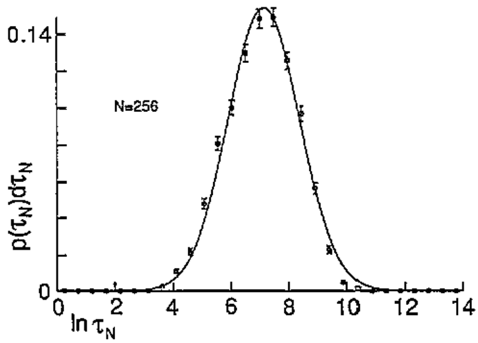

Figure 6. The log-normal distribution for the trapping times found from our simulations at $\eta = 0.6$ for $N = 256$ is presented. The circles with error bars represent the numerical results obtained from a histogram. The solid line is the fit with (4), for $\tau_0(N) = 1275$ and $\Delta_N = 1.25$.

and the time $\tau_0(N) \approx 1275$ obtained from our results for $N = 256$, if they are fitted with (4) and (5), is roughly obtained if in figure 3 one looks for the time at which the function $\langle C_2(t)\rangle$ assumes the value ≈ 0.9425, which is intermediate between 0.885 and 1.

For the simpler case of *random sequential* dynamics, [4], similar subtleties concerning the interchange of the limits $N \to \infty$ and $t \to \infty$ have been found. However, in that case, if the limit $t \to \infty$ is taken before $N \to \infty$, the system always ends in a metastable state, i.e. a fixed point attractor, for all values of η. Since for random sequential updating cyclic attractors cannot appear, this is what one would 'extrapolate' from our study for that simpler case. However, our systems are more complicated. In particular, due to the lack of cyclic attractors in the case of [4], the different dynamic behaviour seen by us with parallel dynamics for $\eta < 0.5$ and > 0.5, respectively, with the possibility of extreme long cycles in the first-mentioned case [3], but only fixed-point and 2-cycles in the second case, has no correspondence at all in [4].

In conclusion, we have shown that the dynamic behaviour of Sherrington–Kirkpatrick spin glasses with non-symmetric coupling under parallel updating is subtle, such that for symmetry parameters $\eta > 0.5$ and $T = 0$ there are at least three characteristic times, namely (i) the short *mean-field* time $\tau_{\rm MF}$ (≈ 100 steps), which determines the approach to the plateau values $\langle C_2(t = 100)\rangle = C_{\rm MF}$ plotted in figure 2 and does not depend on N, (ii) the *trapping* time τ_N, which is much larger and increases exponentially with N, e.g. to values as large as 20000 for $N = 256$ obtained with $\eta = 0.6$ and $T = 0$ (see figure 3), and (iii) the 'transient time' $\tau_0(N)$ appearing in the 'log-normal distribution' of (5) and (6). This time, too, seems to increase exponentially with N and can be roughly characterized as that time where for given N the correlation function $\langle C_2(t)\rangle$ assumes the intermediate value between $C_{\rm MF}$ and 1. For times larger than τ_N, the system is trapped in a 2-cycle or a fixed point for $\eta > 0.5$, whereas for $\eta < 0.5$ also extremely long cycles form, as already found in [3]. However, the trapping phenomenon is suppressed already by small thermal noise.

At the end, the question remains as to what physical mechanisms lead to the above-mentioned trapping and the related times $\tau_0(N)$ and τ_N. There seems to be some kind of noise related to the non-symmetry, i.e. this noise would *increase* with decreasing symmetry η. Partly this noise seems to act like thermal noise, i.e. it tends to suppress $\langle C_2(t)\rangle$. However, for $\eta > 0.5$ with deterministic dynamics, i.e. for $T = 0$, the tendency of the system to get trapped in a fixed point or 2-cycle seems to win, although only at times, which increase exponentially with the system size. Finally, in the same case, but for $\eta < 0.5$, also cycles with exponentially long periods appear, see [3].

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